Fixed-point elimination in the Intuitionistic Propositional Calculus

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FIXED-POINT ELIMINATION IN THE INTUITIONISTIC PROPOSITIONAL CALCULUS

SILVIO GHILARDI, MARIA JOÃO GOUVEIA, AND LUIGI SANTOCANALE

Abstract. It is a consequence of existing literature that least and greatest fixed-points of monotone polynomials on Heyting algebras—that is, the algebraic models of the Intuitionistic Propositional Calculus—always exist, even when these algebras are not complete as lattices. The reason is that these extremal fixed-points are definable by formulas of the **IPC**. Consequently, the μ-calculus based on intuitionistic logic is trivial, every μ-formula being equivalent to a fixed-point free formula. We give in this paper an axiomatization of least and greatest fixed-points of formulas, and an algorithm to compute a fixed-point free formula equivalent to a given μ-formula. The axiomatization of the greatest fixed-point is simple. The axiomatization of the least fixed-point is more complex, in particular every monotone formula converges to its least fixed-point by Kleene's iteration in a finite number of steps, but there is no uniform upper bound on the number of iterations. We extract, out of the algorithm, upper bounds for such n, depending on the size of the formula. For some formulas, we show that these upper bounds are polynomial and optimal.

1. Introduction

In [23] the author proved that, for each formula $\phi(x)$ of the Intuitionistic Propositional Calculus, there exists a number $n \geq 0$ such that $\phi^n(x)$—the formula obtained from ϕ by iterating n times substitution of ϕ for the variable x—and $\phi^{n+2}(x)$ are equivalent in Intuitionistic Logic. This result has, as an immediate corollary, that a syntactically monotone formula $\phi(x)$ converges both to its least fixed-point and to its greatest fixed-point in at most n steps. Using a modern notation based on μ-calculi [3], we have $\mu_x.\phi(x) = \phi^n(\bot)$ and $\nu_x.\phi(x) = \phi^n(\top)$. These identities also show that a μ-calculus based on Intuitionistic Logic is trivial, every μ-formula being equivalent to a fixed-point free formula.

Ruitenberg's work [23] leaves open how to extract or estimate the least number $\rho(\phi)$ such that $\phi^{\rho(\phi)}(x) = \phi^{\rho(\phi)+2}(x)$. Yet, our motivations stem from the theory of extremal fixed-points and μ-calculi [3]. In principle, being able to compute or bound Ruitenberg's number $\rho(\phi)$ might end up in an over-approximation of the closure ordinal of ϕ—the least k such that $\mu_x.\phi(x) = \phi^k(\bot)$. For the analogous problem with the greatest fixed-point, we shall see that the least number k such that $\nu_x.\phi(x) = \phi^k(\top)$ is bounded by 1, while $\rho(\phi)$ might be arbitrarily large.

Later in [20], the author gave an independent proof that least fixed-points of monotone formulas are definable within Intuitionistic Logic. His proof relies on semantics methods and on the coding of Intuitionistic Logic into Grzegorczyk's Logic; the proof was further refined in [21] to encompass the standard coding of Intuitionistic Logic into its modal companion, the logic **S4**. Curiously, no mention of greatest fixed-points appears in these works.

Another relevant source for this paper stem from the discovery that **IPC** has uniform interpolants [22], often named *bisimulation quantifiers*. Together with the deduction property of **IPC**, they give the category of (finitely generated) Heyting algebras—that is, the algebraic models of the Intuitionistic Propositional Calculus—a rather strong structure, axiomatized and studied in [14, 15]. It is possible to argue that in every category with similar properties the extremal fixed-points of monotone formulas are definable. This is possible by using quantified formulas analogous to the one used in [9, §3] to argue that **PDL** lacks the uniform interpolation property. In this paper we exploit this idea and the existential bisimulation quantifiers to characterize greatest fixed-points in the Intuitionistic Propositional Calculus.

A μ-calculus is a prototypical kind of computational logic, obtained from a base logic or algebraic system by addition of distinct forms of iteration so to increase expressivity. This paper is part of a line of research whose goal is to understand, under a unified perspective, why alternation-depth hierarchies of μ-calculi are degenerate or trivial. A μ-calculus adds to an underlying logical-algebraic system formal lgfps of formula-terms whose semantic monotonicity can be witnessed at the syntactic level. When addition of extremal fixed-points is iterated, formula-terms with nested extremal fixed-points are generated. The alternation-depth hierarchy [3, §2.6] of a μ-calculus measures the complexity of a formula-term as a function of the nesting of the different types of fixed-points, with respect to a fixed class of models. It is well known that fixed-points that are unguarded can be eliminated in the propositional modal μ-calculus [18]. We can rephrase this fact by saying that the alternation-depth hierarchy of the μ-calculus over distributive lattices is trivial, every μ-term being equivalent to a fixed-point free term. A goal of [12] was to understand closely this result and to generalize it. We were able to exhibit equational classes of lattices $\mathcal{D}_n$—with $\mathcal{D}_0$ the class of distributive lattices—where the extremal fixed-points can be uniformly computed by iterating a formula-term $n+1$ times from the bottom/top of the lattice; moreover, we showed that these uniform upper bounds are optimal. The reasons for the degeneracy of the hierarchy can be ultimately found in the structural theory of lattices.

As we show in this paper, the situation is quite different when the base for the μ-calculus is Intuitionistic Logic, with its standard models the Heyting algebras. Several ingredients contribute to the existence of a closure ordinal of each formula and to its finiteness. Among them, *strongness* of the monotone polynomials on Heyting algebras. This means that a monotone polynomial $\mathfrak{f} : H \longrightarrow H$ over a Heyting algebra H can be considered as a functor enriched over H, when H is consider as a closed category [16]. For some polynomials, existence and finiteness of the closure ordinal is a consequence of being *inflating* (or expanding) and, on the syntactic level, to a restriction to the use of conjunction that determines a notion of disjunctive formula. As far as the greatest fixed-point is concerned, monotone formulas uniformly converge to it after one step. A key ingredient of the algorithm we present is creation of least fixed-points via the *Rolling* equation (cf. Lemma 1), a fact already used in [10]. For Intuitionistic Logic and Heyting algebras, where formula-terms can be semantically antitone (i.e. contravariant), existing greatest fixed-points create least fixed-points. The most striking difference with the case of distributive lattices (and with the case of the varieties $\mathcal{D}_n$) is the absence of a finite uniform upper bound on the closure ordinals, the rate of convergence to the least fixed-point crucially depending on the shape of the formula.

As emphasized in [19] for the propositional modal μ-calculus, once a formula is known to be equivalent to some other formula of smaller complexity, we should also be able to effectively compute this second formula. Thus, the fact that the alternation hierarchy is trivial for μ-calculi based on the **IPC** should not be the end of the story. The main contribution of this paper is to achieve an effective transformation of an intuitionisitc μ-formula into an equivalent fixed-point free intuitionisitc formula. The size of the formula might exhibit an exponential grow during this transformation. Yet, this is mainly due, as usual, to the need of precompiling a formula into an equivalent one in some kind of conjunctive normal form. We might use sharing in substitutions—or introduce the appropriate formalism for approximants to least fixed-points—so that, if we are given an already precompiled formula, then its least fixed-point w.r.t. the variable x is only polynomially bigger than the original formula. For these formulas, we instantiate this claim by explicitly giving a way of computing $f(\phi)$ such that $\mu_x.\phi(x) = \phi^{f(\phi)}(\bot)$, so that $f(\phi)$ is an upper bound to the closure ordinal of ϕ. In some cases we are able to show that $f(\phi)$ is optimal, by exhibiting some formula $\phi(x)$ such that $\phi^{f(\phi)-1}(\bot) < \mu_x.\phi(x)$.

The paper is structured as follows. We recall in Section 2 some elementary facts from fixed-point theory. In Section 3 we recall the Intuitionistic Propositional Calculus and introduce the Intuitionistic Propositional μ-Calculus. In Section 4 we argue that monotone polynomials are strong and exhibit the interactions between least fixed-points and strong functions. In Section 5 we use the existential bisimulation quantifier to argue that monotone polynomials converge to their greatest fixed-point in one step. Section 6 is the core of our paper, where we show ho to eliminate a least fixed-point from a formula. Together with the result in the previous Section, this leads to a procedure to eliminate off the fixed-points from a $\mathbf{IPC}_\mu$ formula. Finally, in Section 7, we show how upper bounds to closure ordinals can be extracted from the procedure elimination of the least fixed-points. In Section 8 we present our final remarks.

2. Notation and elementary concepts

Let P and Q be posets. A function $f : P \longrightarrow Q$ is *monotone* if $x \leq y$ implies $f(x) \leq f(y)$, for each $x, y \in P$. If $f : P \longrightarrow P$ is a monotone endofunction, then $x \in P$ is a *prefixed-point* of f if $f(x) \leq x$; we denote by $\mathtt{Pre}_f$ the set of prefixed points of f. Whenever $\mathtt{Pre}_f$ has a least element, we denote it by $\mu.f$. Therefore, $\mu.f$ denotes the *least prefixed-point* of f, whenever it exists. If $\mu.f$ exists, then it is a fixed-point of f, necessarily the least one. The notions of least prefixed-point and of least fixed-point coincide on complete lattices or when the least fixed-point is computed by iterating from the bottom of a lattice; for our purposes they are interchangeable, so we shall abuse of language and refer to $\mu.f$ as the least fixed-point of f. Dually (and abusing again of language), the *greatest fixed-point* of f shall be denoted by $\nu.f$.

Let us mention few elementary facts from fixed-point theory.

Lemma 1. *Let P, Q be posets, $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ be monotone functions. If $\mu.(g \circ f)$ exists, then $\mu.(f \circ g)$ exists as well and is equal to $f(\mu.(g \circ f))$.*

As we do not work in complete lattices (so we are not ensured that least fixed-points exist) we express the above statement via the equality

$$\mu.(f \circ g) := f(\mu.(g \circ f)) \,, \tag{Roll}$$

where the colon emphasizes existence: if the least fixed-point in the expression on the right *exists*, then this expression is the least fixed-point of $f \circ g$. Analogous notations will be used later. We endow the product of two posets P and Q with the coordinatewise ordering. Therefore a function $f : P \times Q \longrightarrow R$ is monotone if, as a function of two variables, it is monotone in each variable. To deal with least fixed-points of functions of many variables, we use the standard notation: for example, if $f : P \times P \longrightarrow P$ is the monotone function $f(x, y)$, then, for a fixed $p \in P$, $\mu_x.f(x, p)$ denotes the least fixed-point of $f(x, p)$. Let us recall that the correspondence $p \mapsto \mu_x.f(x, p)$—noted $\mu_x.f(x, y)$—is again monotone.

Lemma 2. *If P is a poset and $f : P \times P \longrightarrow P$ is a monotone mapping, then*

$$\mu_x.f(x, x) := \mu_x.\mu_y.f(x, y) \,. \tag{Diag}$$

Lemma 3. *If P and Q are posets and $\langle f, g\rangle : P \times Q \longrightarrow P \times Q$ is a monotone function, then $\mu.\langle f, g\rangle := \langle \mu_1, \mu_2\rangle$, where*

$$\mu_1 = \mu_x.f(x, \mu_y.g(x, y)) \quad \textit{and} \quad \mu_2 = \mu_y.g(\mu_1, y) \,. \tag{Bekic}$$

3. The Intuitionistic Propositional μ-Calculus

Formulas of the Intuitionistic Propositional Calculus are generated according to the following grammar:

$$\phi \;\; \Rightarrow \;\; x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \phi \to \phi \,, \tag{1}$$

where x ranges over a countable set $\mathbb{X}$ of propositional variables. For the **IPC**, the formulation of the consequence relation $\vdash_{\mathbf{LJ}}$ (relating a set of formulas to a formula) goes back to Gentzen's work on the system **LJ** [13]. It is well known that Intuitionistic Logic is sound and complete w.r.t. the class of its algebraic models, the Heyting algebras.

Definition 1. A *Heyting algebra* H is a bounded lattice (with least element $\bot$ and greatest element $\top$) equipped with a binary operation $\to$ such that the following equations hold in H:

$$\begin{aligned} x \wedge (x \to y) &= x \wedge y \,, & x \wedge (y \to x) &= x \,, \\ x \to x &= \top \,, & x \to (y \wedge z) &= (x \to y) \wedge (x \to z) \,. \end{aligned} \tag{2}$$

We can define on any Heyting algebra a partial order by saying that $x \leq y$ holds when $x \vee y = y$. We identify formulas of the **IPC** with terms of the theory of Heyting algebras, constructed therefore from variables and using the signature $\langle \top, \wedge, \bot, \vee, \to\rangle$. For ϕ such a formula-term, H a Heyting algebra, and $v : \mathbb{X} \longrightarrow H$ a valuation of the propositional variables in H, let us write $[\![\phi]\!]_v$ for the result of evaluating the formula in H, starting from the variables. The soundness and completeness theorem of the **IPC** over Heyting algebras—see e.g. [6]—can then be stated as follows: *if Γ is a finite set of formula-terms and ϕ is a formula-term, then $\Gamma \vdash_{\mathbf{LJ}} \phi$ holds if and only if $\bigwedge_{\gamma \in \Gamma} [\![\gamma]\!]_v \leq [\![\phi]\!]_v$ holds, in every Heyting algebra H and for every valuation of the propositional variables $v : \mathbb{X} \longrightarrow H$.* Given this theorem, we shall often abuse of notation and write $\leq$ in place of $\vdash_{\mathbf{LJ}}$, and the equality symbol $=$ to denote logical equivalence of formulas.

We aim at studying extremal fixed-points on Heyting algebras. To this end, we formalize the Intuitionistic Propositional μ-Calculus.

An occurrence of a variable x is *positive* in a formula-term ϕ if, in the syntax tree of ϕ, the path from the root to the leaf labeled by this variable contains an even number of nodes labeled by subformulas $\psi_1 \to \psi_2$ immediately followed by a node labeled by the subformula ψ_1. If, on this path the number of those nodes is odd, then we say that this occurrence of x is *negative* in ϕ. A variable x is positive in a formula ϕ if each occurrence of x is positive in ϕ. A variable x is negative in a formula ϕ if each occurrence of x is negative in ϕ. If we add to the previous grammar (1) the following productions:

$$\phi \;\Rightarrow\; \mu_x.\phi\,, \qquad\qquad \phi \;\Rightarrow\; \nu_x.\phi\,,$$

subject to the restriction that x is positive in ϕ, we obtain then a grammar for the formulas of $\mathbf{IPC}_\mu$, the Intuitionistic Propositional μ-Calculus. The semantics of these formulas is the expected one. Let ϕ be a formula of $\mathbf{IPC}_\mu$, and let x be positive in ϕ. Let us denote by $\mathbb{X}_\phi$ the set of variables having an occurrence in ϕ. If $v : \mathbb{X}_\phi \setminus \{\, x \,\} \longrightarrow H$ is a valuation of all the variables of ϕ but x in a *complete* Heyting algebra, then the function $[\![\phi]\!]_v$, defined by

$$h \mapsto [\![\phi]\!]_{v,h/x}\,,$$

is monotone, so $\mu_x.\phi$ (resp., $\nu_x.\phi$) is to be evaluated over the least fixed-point (resp., the greatest fixed-point) of this function. A sequent calculus for $\mathbf{IPC}_\mu$ is presented in [7, §2].

Let us say that a formula ϕ of $\mathbf{IPC}_\mu$ is fixed-point free if it is a formula of **IPC**, that is, it does not contain either of the symbols μ, ν.

Proposition 4. *Every formula ϕ of $\mathbf{IPC}_\mu$ is equivalent to a fixed-point free formula ϕ'.*

Proof. Clearly, the statement holds if we can show that it holds whenever $\phi = \mu_x.\psi$ or $\phi = \nu_x.\psi$, where ψ is a fixed-point free formula. For a natural number $n \geq 0$, let $\psi^n(x)$ denote the formula obtained by substituting x for ψ n times. Ruitenburg [23] proves that, for each intuitionisitic formula ψ, there exists a number $n \geq 0$ such that the formulas $\psi^n(x)$ and $\psi^{n+2}(x)$ are equivalent. If x is positive in ψ, then instantiating x with $\bot$, leads to the equivalence $\psi^{n+1}(\bot) \equiv \psi^n(\bot)$, exhibiting $\psi^n(\bot)$ as the least fixed-point of ψ. Similarly, $\psi^n(\top)$ is the greatest fixed-point of ψ. $\square$

While it is an obvious step to derive the previous Proposition from Ruitenburg's result, there has been no attempt (as far as we know) to compute an upper bound on $n \geq 0$ such that $\psi^n(x)$ and $\psi^{n+2}(x)$ are equivalent. Nor is such an n necessarily a tight upper bound for convergence of a formula to its least or greatest fixed-point.

4. Strong monotone functions and fixed-points

If H is a Heyting algebra and $f : H \longrightarrow H$ is a monotone function, then we say that f is *strong* if

$$x \wedge f(y) \leq f(x \wedge y)\,, \qquad \text{for any } x, y \in H.$$

The interplay between fixed-points and this class of functions has already been emphasized, mainly in the context of categorical proof-theory and semantics of functional programming languages with inductive data types [8, 7].

Lemma 5. *A monotone $f : H \longrightarrow H$ is strong if and only if any of the following equivalent conditions holds in H:*

$$x \wedge f(y) \leq f(x \wedge y)\,, \tag{3}$$
$$f(x \to y) \leq x \to f(y)\,, \tag{4}$$
$$x \to y \leq f(x) \to f(y)\,. \tag{5}$$

The proof of these equivalences is usual in categorical algebra [17] and therefore it is omitted here.

Definition 2. Let H be a Heyting algebra. We say that a function $f : H \longrightarrow H$ is *monotone polynomial* if there exist a formula ϕ of the **IPC**, a variable x positive in ϕ, and a valuation $\vec{v} : \mathbb{X}_\phi \setminus \{\, x \,\} \longrightarrow H$ such that, for each $h \in H$, we have $f(h) = [\![\phi]\!]_{\vec{v},h/x}$.

Proposition 6. *Every monotone polynomial $\mathfrak{f}$ on a Heyting algebra is strong.*

Proof. Recall that the replacement Lemma holds in the **IPC**: $z \leftrightarrow w \vdash_{\mathbf{LJ}} \phi(z) \leftrightarrow \phi(w)$. Substituting x for z and $x \wedge y$ for w, and considering that $x \to y \vdash_{\mathbf{LJ}} x \leftrightarrow (x \wedge y)$, we derive that $x \to y \vdash_{\mathbf{LJ}} \phi(x) \leftrightarrow \phi(x \wedge y)$. Assuming that u is positive in $\phi(u)$, we have $\phi(x) \leftrightarrow \phi(x \wedge y) \vdash_{\mathbf{LJ}} \phi(x) \to \phi(x \wedge y) \vdash_{\mathbf{LJ}} \phi(x) \to \phi(y)$, whence $x \to y \vdash_{\mathbf{LJ}} \phi(x) \to \phi(y)$. The last relation immediately implies that equation (5) from Lemma 5 holds, when $\mathfrak{f}$ is a monotone polynomial. □

It can be shown that the relation $\mathfrak{f}(x) \wedge y = \mathfrak{f}(x \wedge y) \wedge y$ holds (for any x, y and) for *any* polynomial on a Heyting algebra. The analogous remark for Boolean algebras is credited to Peirce, in view of the iteration rule for existential graphs of type Alpha, see [11].

Proposition 7. *If $\mathfrak{f}$ is a strong monotone function on H and $a \in H$, then*

$$\mu.a \to \mathfrak{f} := a \to \mu.\mathfrak{f}\,, \qquad\qquad \mu.a \wedge \mathfrak{f} := a \wedge \mu.\mathfrak{f}\,. \tag{6}$$

Proof. Let us argue first that first equation holds. To this end, let us set $\mathfrak{f}^a(x) =_{\mathtt{def}} a \to \mathfrak{f}(x)$. From $\mathfrak{f} \leq \mathfrak{f}^a$ we have $\mathtt{Pre}_{\mathfrak{f}^a} \subseteq \mathtt{Pre}_{\mathfrak{f}}$. Thus, if $p \in \mathtt{Pre}_{\mathfrak{f}^a}$, then $\mu_x.\mathfrak{f}(x) = \mathfrak{f}(\mu_x.\mathfrak{f}(x)) \leq \mathfrak{f}(p)$ and $a \to \mu.\mathfrak{f} \leq a \to \mathfrak{f}(p) = \mathfrak{f}^a(p) \leq p$. That is, $a \to \mu.\mathfrak{f}$ is below any element of $\mathtt{Pre}_{\mathfrak{f}^a}$. To obtain the proposition, we need to argue that $a \to \mu.\mathfrak{f}$ belongs to $\mathtt{Pre}_{\mathfrak{f}^a}$. To this end, we notice that $\{\, a \to p \mid p \in \mathtt{Pre}_{\mathfrak{f}} \,\} \subseteq \mathtt{Pre}_{\mathfrak{f}^a}$, since if $\mathfrak{f}(p) \leq p$, then $\mathfrak{f}^a(a \to p) = a \to \mathfrak{f}(a \to p) \leq a \to \mathfrak{f}(p) \leq a \to p$, where we used the fact that $\mathfrak{f}$ is strong, thus (4) holds.

Let us come now to the second equation, for which we set $\mathfrak{f}_a(x) =_{\mathtt{def}} a \wedge \mathfrak{f}(x)$. Suppose $a \wedge \mathfrak{f}(p) \leq p$, so $\mathfrak{f}(p) \leq a \to p$. Then $\mathfrak{f}(a \to p) \leq a \to \mathfrak{f}(p) \leq a \to p$, using (4), whence $\mu.\mathfrak{f} \leq a \to p$ and $a \wedge \mu.\mathfrak{f} \leq p$. Thus we are left to argue that $a \wedge \mu.\mathfrak{f}$ is a prefixed-point of $\mathfrak{f}_a$. Yet, this is true for an arbitrary prefixed-point p of $\mathfrak{f}$: $a \wedge \mathfrak{f}(a \wedge p) \leq a \wedge \mathfrak{f}(p) \leq a \wedge p$. □

Corollary 8. *For each $n \geq 1$ and each collection $\mathfrak{f}_i$, $i = 1, \ldots, n$ of monotone polynomials, we have the following distribution law:*

$$\mu_x.\bigwedge_{i=1,\ldots,n} \mathfrak{f}_i(x) := \bigwedge_{i=1,\ldots,n} \mu_x.\mathfrak{f}_i(x)\,. \tag{7}$$

Proof. For $n = 1$ there is nothing to prove. We suppose therefore that the statement holds for every collection of size $n \geq 1$, and prove it holds for a collection of size $n + 1$. We have

$$\begin{aligned}
\mu_x.(\mathfrak{f}_{n+1}(x) \wedge \bigwedge_{i=1,\dots,n} \mathfrak{f}_i(x)) &:= \mu_x.\mu_y.(\mathfrak{f}_{n+1}(y) \wedge \bigwedge_{i=1,\dots,n} \mathfrak{f}_i(x)), && \text{by } (\mathtt{Diag}),\\
&:= \mu_x.((\mu_y.\mathfrak{f}_{n+1}(y)) \wedge \bigwedge_{i=1,\dots,n} \mathfrak{f}_i(x)), && \text{by (6)},\\
&:= (\mu_y.\mathfrak{f}_{n+1}(y)) \wedge \mu_x.(\bigwedge_{i=1,\dots,n} \mathfrak{f}_i(x)), && \text{again by (6)},\\
&:= (\mu_y.\mathfrak{f}_{n+1}(y)) \wedge \bigwedge_{i=1,\dots,n} \mu_x.\mathfrak{f}_i(x), && \text{by the IH.} \quad \square
\end{aligned}$$

The elimination of greatest fixed-points is easy for strong monotone functions (we are thankful to the referee for pointing out the following fact, which greatly simplifies our original argument):

Proposition 9. *If $\mathfrak{f} : L \longrightarrow L$ is any strong monotone function on a bounded lattice L, then $\mathfrak{f}^2(\top) = \mathfrak{f}(\top)$. Thus $\mathfrak{f}(\top)$ is the greatest fixed-point of $\mathfrak{f}$.*

Proof. Indeed, we have $\mathfrak{f}(\top) = \mathfrak{f}(\top) \wedge \mathfrak{f}(\top) \leq \mathfrak{f}(\mathfrak{f}(\top) \wedge \top) = \mathfrak{f}^2(\top)$. $\square$

5. A digression on fixpoints and bisimulation quantifiers

The connection between extremal fixed-points and bisimulation quantifiers, as emphasized in [9], was a main motivation to tackle this research. Although in the end our computations are independant on that, we nevertheless want to have a closer look to the topic (the content of this section is not needed afterwards).

It was discovered in [22] that **IPC** has the uniform interpolation property. As made clear from the title of [22], this property amounts to an internal existential and universal quantification. This result was further refined in [15] to show that any morphism between finitely generated Heyting algebras has a left and a right adjoint. We shall be interested in Heyting algebras $H[x]$ of polynomials with coefficients from H, and to (the left and right adjoints to) the inclusion of H into $H[x]$. The algebra of polynomials $H[x]$ is formally defined as the coproduct (in the category of Heyting algebras) of H with the free Heyting algebra on one generator. The universal property gives that if $h_0 \in H$, then there exists a unique morphism $[\![\cdot]\!]_{h_0/x} : H[x] \longrightarrow H$ such that $[\![x]\!]_{h_0/x} = h_0$ and $[\![h]\!]_{h_0/x} = h$, for each $h \in H$. Thus, for $\mathfrak{f} \in H[x]$ and $h \in H$, we can define $\mathfrak{f}(h) = [\![\mathfrak{f}]\!]_{h/x}$. It follows from [15] that if H is finitely generated, then the inclusion $i_x : H \longrightarrow H[x]$ has both adjoints $\exists_x, \forall_x : H[x] \longrightarrow H$, with $\exists_x \dashv i_x \dashv \forall_x$. In particular, we shall use the unit relation for $\exists_x$:

$$\mathfrak{f} \leq i_x(\exists_x(\mathfrak{f}))\,, \qquad \text{for all } \mathfrak{f} \in H[x]\,.$$

Identifying $h \in H$ with $i_x(h) \in H[x]$, we can read the above inequality as $\mathfrak{f} \leq \exists_x.\mathfrak{f}$. We can identify a monotone polynomial, as defined in Definition 2, as an element $\mathfrak{f} \in H[x]$ such that $[\![\mathfrak{f}]\!]_{h_0/x} \leq [\![\mathfrak{f}]\!]_{h_1/x}$ whenever $h_0 \leq h_1$.

Proposition 10. *If $\mathfrak{f}$ is a monotone polynomial on a finitely generated Heyting algebra, then*

$$\nu.\mathfrak{f} := \exists_x.(x \wedge (x \to \mathfrak{f}(x))) \,. \tag{8}$$

Proof. By the unit relation $x \wedge x \to \mathfrak{f}(x) \leq \exists_x.(x \wedge x \to \mathfrak{f}(x))$. Recall that evaluation at $p \in H$ is a Heyting algebra morphism, thus it is monotone. Therefore, if $p \in H$ is a postfixed-point of $\mathfrak{f}$, then by evaluating the previous inequality at p, we have

$$p = p \wedge p \to \mathfrak{f}(p) \leq \exists_x.(x \wedge x \to \mathfrak{f}(x)) \,,$$

so that $\exists_x.(x \wedge x \to \mathfrak{f}(x))$ is greater than any postfixed-point of $\mathfrak{f}$. Let us show that $\exists_x.(x \wedge x \to \mathfrak{f}(x))$ is also a postfixed-point. To this end, it will be enough to argue that $x \wedge x \to \mathfrak{f}(x) \leq \mathfrak{f}(\exists_x.(x \wedge x \to \mathfrak{f}(x)))$ in $H[x]$. We compute as follows:

$$\begin{aligned} x \wedge x \to \mathfrak{f}(x) &\leq \mathfrak{f}(x) \wedge x \to \mathfrak{f}(x) \\ &\leq \mathfrak{f}(x \wedge x \to \mathfrak{f}(x)), && \text{since } \mathfrak{f} \text{ is strong, by (3),} \\ &\leq \mathfrak{f}(\exists_x.(x \wedge x \to \mathfrak{f}(x))), && \text{since } \mathfrak{f} \text{ is monotone.} \quad \square \end{aligned}$$

In a similar fashion, we can prove that if $\mathfrak{f}$ is a monotone polynomial on a finitely generated Heyting algebra, then $\mu.\mathfrak{f} := \forall_x.((\mathfrak{f}(x) \to x) \to x)$. As an application, we give an alternative proof of Proposition 9:

Corollary 11. *If $\mathfrak{f}$ is a monotone polynomial on a Heyting algebra H, then*

$$\nu.\mathfrak{f} := \mathfrak{f}(\top) \,. \tag{9}$$

Proof. It is easy to see that if $\mathfrak{f}$ is a monotone polynomial on a finitely generated Heyting algebra, then $\exists_x.\mathfrak{f} = \mathfrak{f}(\top)$. Thus we have

$$\nu.\mathfrak{f} = \exists_x.(x \wedge (x \to \mathfrak{f}(x))) = \exists_x.(x \wedge \mathfrak{f}(x)) = \top \wedge \mathfrak{f}(\top) = \mathfrak{f}(\top) \,.$$

Therefore, if ϕ is a formula-term whose variables are among set $x, y_1, \ldots, y_n$, then the equation $\phi^2(\top) = \phi(\top)$ holds in the free Heyting algebra on the set $\{ y_1, \ldots, y_n \}$. Consequently, the equation $\mathfrak{f}(\top) = \mathfrak{f}^2(\top)$ holds in H, making $\mathfrak{f}(\top)$ into the greatest fixed-point of $\mathfrak{f}$. $\square$

6. The elimination procedure

In this Section we present our main result, a procedure that both axiomatizes and eliminates least fixed-points of the form $\mu_x.\phi(x)$ with ϕ fixed-point free. Together with the axiomatization of greatest fixed-points given in Section 5, the procedure can be extended to a procedure to construct a fixed-point free formula ψ equivalent to a given formula χ of the $\mathbf{IPC}_\mu$.

Definition 3. An occurrence of the variable x is *strongly positive* in a formula-term ϕ if there is no subformula ψ of ϕ of the form $\psi_0 \to \psi_1$ such that x is located in ψ_0. A formula-term ϕ is *strongly positive* in the variable x if every occurrence of x is *strongly positive* in ϕ. An occurrence of a variable x is *weakly negative* in a formula-term ϕ if it is not strongly positive. A formula-term ϕ is *weakly negative* in the variable x if every occurrence of x is *weakly negative* in ϕ.

Observe that a variable might be neither strongly positive nor weakly negative in a formula-term. A second key concept for the elimination is the following notion of disjunctive formula.

Definition 4. The set of formula-terms that are *disjunctive in the variable* x is generated by the following grammar:

$$\phi \;\; \Rightarrow \;\; x \mid \beta \vee \phi \mid \phi \vee \beta \mid \alpha \to \phi \mid \phi \vee \phi\,, \tag{10}$$

where α and β are formulas with no occurrence of the variable x. A formula-term ϕ is in *normal form* (w.r.t. x) if it is a conjunction of formula-terms ϕ_i, $i \in I$, so that each ϕ_i either does not contain the variable x, or it is disjunctive in x.

Notice that disjunctive formula-terms are strongly positive in x. Due to equation (2) and since the usual distributive laws hold in Heyting algebras, we have the following Lemma.

Lemma 12. *Every strongly positive formula-term is equivalent to a formula-term in normal form.*

In order to compute the least fixed-point $\mu_x.\phi$, we take the following steps:

(1) We rename all the weakly negative occurrences of x in ϕ to a fresh variable y, so $\phi(x) = \psi(x, x/y)$ with ψ strongly positive in x and weakly negative in y.
(2) We compute a normal form of $\psi(x, y)$, so this formula is equivalent to a conjunction $\bigwedge_{i\in I} \psi_i(x, y)$ with each ψ_i disjunctive in x or not containing the variable x.
(3) *Strongly positive elimination.* For each $i \in I$: if x has an occurrence in ψ_i, we compute then a formula ψ'_i equivalent to the least fixed-point $\mu_x.\psi_i(x, y)$ and observe that ψ'_i is weakly negative in y; otherwise, we let $\psi'_i = \psi_i$.
(4) *Weakly negative elimination.* The formula $\bigwedge_{i\in I} \psi'_i(y)$ is weakly negative in y; we compute a formula χ equivalent to $\mu_y. \bigwedge_i \psi'_i(y)$ and return it.

The correction of the procedure relies on the following chain of equivalences:

$$\begin{aligned} \mu_x.\phi(x) &= \mu_y.\mu_x.\psi(x, y) = \mu_y.\mu_x. \bigwedge_{i\in I} \psi_i(x, y), && \text{where we use } (\texttt{Diag}),\\ &= \mu_y. \bigwedge_{i\in I} \mu_x.\psi_i(x, y) = \mu_y. \bigwedge_{i\in I} \psi'_i(y) = \chi, && \text{where we have used (7).} \end{aligned}$$

6.1. **Strongly positive elimination.** We tackle here the problem of computing the least fixed-point $\mu_x.\phi$ of a formula-term ϕ which is *disjunctive* in x. Recall that the formulas α and β appearing in a parse tree as leaves—according to the grammar (10)—do not contain the variable x. We call such a formula α a *head subformula* of ϕ, and such a β a *side subformula* of ϕ, and thus we put:

$$\begin{aligned} \mathsf{Head}(\phi) &=_{\mathtt{def}} \{\, \alpha \mid \alpha \text{ is a head subformula of } \phi \,\}\,,\\ \mathsf{Side}(\phi) &=_{\mathtt{def}} \{\, \beta \mid \beta \text{ is a side subformula of } \phi \,\}\,. \end{aligned}$$

Recall that a monotone function $f : P \longrightarrow P$ is *inflating* if $x \leq f(x)$.

Lemma 13. *The interpretation of a strongly positive disjunctive formula ϕ as a function of x is inflating.*

The key observation needed to prove Proposition 15 is the following Lemma on monotone inflating functions. In the statement of the Lemma we assume that P is a join-semilattice, and that $f \vee g$ is the pointwise join of the two functions f and g.

Lemma 14. *If $f, g : P \longrightarrow P$ are monotone inflating functions, then $\mathtt{Pre}_{f\vee g} = \mathtt{Pre}_{f\circ g}$. Consequently, for any monotone function $h : P \longrightarrow P$, we have*

$$\mu.(\, f \vee g \vee h\,) :=: \mu.(\,(f \circ g) \vee h\,)\,. \tag{11}$$

Proof. Observe firstly that $\mathtt{Pre}_{f\vee g} = \mathtt{Pre}_f \cap \mathtt{Pre}_g$. If $p \in \mathtt{Pre}_{f\circ g}$, then $f(p) \leq f(g(p)) \leq p$ and $g(p) \leq f(g(p)) \leq p$, showing that $p \in \mathtt{Pre}_{f\vee g}$. Conversely, if $p \in \mathtt{Pre}_{f\vee g}$, then p is a fixed point of both f and g, since these functions are inflating. It follows that $f(g(p)) = f(p) = p$, showing $p \in \mathtt{Pre}_{f\circ g}$.

We have argued that $\mathtt{Pre}_{f\vee g}$ coincides with $\mathtt{Pre}_{f\circ g}$; this implies that $\mathtt{Pre}_{(f\circ g)\vee h} = \mathtt{Pre}_{f\vee g\vee h}$ and, from this equality, equation (11) immediately follows. $\square$

To ease reading of the next Proposition and of its proof, let us put

$$[\alpha]\,\phi =_{\mathtt{def}} \alpha \to \phi\,.$$

Proposition 15. *If ϕ is a disjunctive formula-term, then*

$$\mu.\phi = \Bigl[\bigwedge_{\alpha\in\mathsf{Head}(\phi)} \alpha\Bigr]\,(\bigvee_{\beta\in\mathsf{Side}(\phi)} \beta)\,. \tag{12}$$

Proof. For ψ, χ formula-terms, let us write $\psi \sim \chi$ when $\mu.\psi = \mu.\chi$. We say that a disjunctive formula ψ is reduced (w.r.t. ϕ) if either it is x, or it is of the form $\beta \vee x$ (or $x \vee \beta$) for some $\beta \in \mathsf{Side}(\phi)$, or of the form $[\alpha]\,x$ for some $\alpha \in \mathsf{Head}(\phi)$. A set Φ of disjunctive formulas is *reduced* if every formula in Φ is reduced.

We shall compute a reduced set of disjunctive formulas Φ_k such that $\phi \sim \bigvee \Phi_k$. Thus let $\Phi_0 = \{\,\phi\,\}$. If Φ_i is not reduced, then there is $\phi_0 \in \Phi_i$ which is not reduced, thus of the form (a) $\beta \vee \psi$ (or $\psi \vee \beta$) with $\psi \neq x$, or (b) $[\alpha]\,\psi$ with $\psi \neq x$, or (c) $\psi_1 \vee \psi_2$. According to the case (ℓ), with $\ell \in \{\,a, b, c\,\}$, we let Φ_{i+1} be $(\Phi_i \setminus \{\,\phi_0\,\}) \cup \Psi_\ell$ where Ψ_ℓ is as follows:

$$\Psi_a = \{\,\beta \vee x, \psi\,\}, \qquad \Psi_b = \{\,[\alpha]\,x, \psi\,\}, \qquad \Psi_c = \{\,\psi_1, \psi_2\,\}\,.$$

By Lemma 14, we have $\bigvee \Phi_i \sim \bigvee \Phi_{i+1}$. Morever, for some $k \geq 0$, Φ_k is reduced and $\Phi_k \subseteq \{\,[\alpha]\,x \mid \alpha \in \mathsf{Head}(\phi)\,\} \cup \{\,\beta \vee x \mid \beta \in \mathsf{Side}(\phi)\,\} \cup \{\,x\,\}$. Consequently

$$\mu_x.\phi(x) = \mu_x.\bigvee \Phi_k \leq \mu_x.(x \vee \bigvee_{\alpha\in\mathsf{Head}(\phi)} [\alpha]\,x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x)\,. \tag{13}$$

On the other hand, if $\alpha \in \mathsf{Head}(\phi)$, then $\phi(x) = \psi_1(x, [\alpha]\,\psi_2(x))$ for some disjunctive formulas ψ_1 and ψ_2, so

$$[\alpha]\,x \leq [\alpha]\,\psi_2(x) \leq \psi_1(x, [\alpha]\,\psi_2(x)) = \phi(x)$$

and, similarly, $\beta \vee x \leq \phi(x)$, whenever $\beta \in \mathsf{Side}(\phi)$. It follows that

$$x \vee \bigvee_{\alpha\in\mathsf{Head}(\phi)} [\alpha]\,x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x \leq \phi(x)\,,$$

whence, by taking the least fixed-point in both sides of the above inequality, we derive equality in (13). Finally, in order to obtain (12), we compute as follows:

$$
\begin{aligned}
&\mu_x.(x \vee \textstyle\bigvee_{\alpha\in\mathsf{Head}(\phi)} [\alpha]\, x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x) \\
&\quad = \mu_x.([\alpha_1]\ldots[\alpha_n]\, x \vee (x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x)) \\
&\qquad\qquad \text{by Lemma 14, with } \mathsf{Head}(\phi) = \{\, \alpha_1, \ldots, \alpha_n \,\}, \\
&\quad = \mu_x.(\left[\bigwedge_{\alpha\in\mathsf{Head}(\phi)} \alpha\right] x \vee (x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x)), \\
&\qquad\qquad \text{since } [\alpha_1]\ldots[\alpha_n]\, x = \left[\textstyle\bigwedge_{i=1,\ldots,n} \alpha_i\right] x, \\
&\quad = \mu_x.(\left[\bigwedge_{\alpha\in\mathsf{Head}(\phi)} \alpha\right] (x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x)), \qquad \text{by Lemma 14,} \\
&\quad = \left[\bigwedge_{\alpha\in\mathsf{Head}(\phi)} \alpha\right] \mu_x.(x \vee \bigvee_{\beta\in\mathsf{Side}(\phi)} \beta \vee x), \qquad \text{by Proposition 7,} \\
&\quad = \left[\bigwedge_{\alpha\in\mathsf{Head}(\phi)} \alpha\right] (\bigvee_{\beta\in\mathsf{Side}(\phi)} \beta)\,.
\end{aligned}
$$

6.2. **Weakly negative elimination.** If ϕ is weakly negative in x then we can write

$$\phi(x) = \psi_0(\psi_1(x), \ldots, \psi_n(x))\,, \tag{14}$$

for formula-terms $\psi_0(y_1, \ldots, y_n)$ and $\psi_i(x)$, $i = 1, \ldots, n$, such that: (a) all the variables y_i are negative in ψ_0; (b) for $i = 1, \ldots, n$, x is negative ψ_i.

Proposition 16. *Let $\langle \nu_1, \ldots, \nu_n \rangle$ be a collection of formula-terms denoting the greatest solution of the system of equations $\{\, y_i = \psi_i(\psi_0(y_1, \ldots, y_n)) \mid i = 1, \ldots, n \,\}$. Then $\psi_0(\nu_1, \ldots, \nu_n)$ is a formula equivalent to $\mu_x.\phi(x)$.*

Proof. Let $v : \mathbb{X} \setminus \{\, x, y_1, \ldots, y_n \,\} \longrightarrow H$ be a partial valuation into an Heyting algebra H, put $\mathfrak{f}_0 = [\![\psi_0]\!]_v$ and, for $i = 1, \ldots, n$, $\mathfrak{f}_i = [\![\psi_i]\!]_v$. Then $\mathfrak{f}_0$ is a monotone function from $[H^{op}]^n$ to H. Here H^{op} is the poset with the same elements as H but with the opposite ordering relation. Similarly, for $1 \leq i \leq n$, $\mathfrak{f}_i : H \longrightarrow H^{op}$. If we let $\bar{\mathfrak{f}} = \langle \mathfrak{f}_i \mid i = 1, \ldots, n \rangle \circ \mathfrak{f}_0$, then $\bar{\mathfrak{f}} : [H^{op}]^n \longrightarrow [H^{op}]^n$. We exploit next the fact that $(\cdot)^{op}$ is a functor, so that $f^{op} : P^{op} \longrightarrow Q^{op}$ is the same monotone function as f, but considered as having distinct domain and codomain. Then, using (`Roll`), we can write

$$
\begin{aligned}
\mu.(\, \mathfrak{f}_0 \circ \langle \mathfrak{f}_i \mid i = 1, \ldots, n \rangle \,) &= \mathfrak{f}_0(\, \langle \mathfrak{f}_i \mid i = 1, \ldots, n \rangle \circ \mathfrak{f}_0 \,) \\
&= \mathfrak{f}_0(\, \mu.\bar{\mathfrak{f}} \,) = \mathfrak{f}_0(\, \nu.\bar{f}^{op} \,)\,,
\end{aligned} \tag{15}
$$

since the least fixed-point of f in P^{op} is the greatest fixed-point of f^{op} in P. That is, if we consider the function $\langle \mathfrak{f}_i \mid i = 1, \ldots, n \rangle \circ \mathfrak{f}_0$ as sending a tuple of elements of H (as opposite to H^{op}) to another such a tuple, then equation (15) proves that a formula denoting the least fixed-point of ϕ is constructible out of formulas for the greatest solution of the system mentioned in the statement of the Proposition. $\square$

As far as computing the greatest solution of the system mentioned in the Proposition, this can be achieved by using the Bekic elimination principle, see Lemma 3. This principle implies that solutions of systems can be constructed from solutions of linear systems, i.e. from usual parametrized fixed-points. In our case, as witnessed by equation (9), these parametrized greatest fixed-points are computed by substituting $\top$ for the fixed-point variable. In the next Section we shall give a more explicit description, by means of approximants, of the least fixed-point of a weakly negative formula ϕ.

7. Upper bounds on closure ordinals

Recall that Ruitenburg's result [23] implies that a monotone formula converges to its (parametrized) least fixed-point by iterating the formula n times from $\bot$, for some $n \geq 0$. That is, we can always substitute $\mu_x.\phi(x)$ for some equivalent $\phi^n(\bot)$. We show, in this Section, how to extract, from the procedure just seen, upper bounds for such a number n.

Proposition 17. *If ϕ is a disjunctive formula and n is the cardinality of the set $\mathsf{Head}(\phi)$, then*

$$\mu_x.\phi(x) = \phi^{n+1}(\bot)\,. \tag{16}$$

Proof. We have seen, in the proof of Propositon 15, that $[\alpha]\,x \leq \phi(x)$ for any $\alpha \in \mathsf{Head}(\phi)$ and, similarly, $\beta \vee x \leq \phi(x)$ for any $\beta \in \mathsf{Side}(\phi)$. Thus we have

$$\bigvee_{\beta \in \mathsf{Side}(\phi)} \beta = \bigvee_{\beta \in \mathsf{Side}(\phi)} \beta \vee \bot \leq \phi(\bot)\,.$$

Let $\mathsf{Head}(\phi) = \{\, \alpha_1, \ldots, \alpha_n \,\}$. Supposing that $[\alpha_i] \ldots [\alpha_1]\,(\bigvee_{\beta \in \mathsf{Side}(\phi)} \beta) \leq \phi^{i+1}(\bot)$, then

$$[\alpha_{i+1}]\,[\alpha_i] \ldots [\alpha_1]\,(\bigvee_{\beta \in \mathsf{Side}(\phi)} \beta) \leq [\alpha_{i+1}]\,(\phi^{i+1}(\bot)) \leq \phi(\phi^{i+1}(\bot)) = \phi^{i+2}(\bot)\,.$$

Whence

$$\mu_x.\phi(x) = \left[\bigwedge_{i=1,\ldots,n} \alpha_i\right](\bigvee_{\beta \in \mathsf{Side}(\phi)} \beta) = [\alpha_n] \ldots [\alpha_1]\,(\bigvee_{\beta \in \mathsf{Side}(\phi)} \beta) \leq \phi^{n+1}(\bot)\,.$$

The upper bound given in (16) is optimal: if we let $\phi_n(x) =_{\mathtt{def}} b \vee \bigvee_{i=1,\ldots,n} a_i \to x$ and consider the Heyting algebra of downsets of $\langle P(\{\,1,\ldots,n\,\}), \subseteq \rangle$, then, interpreting b as $\{\,\emptyset\,\}$ and a_i as $\{\, s \subseteq \{\,1,\ldots,n\,\} \mid i \notin s \,\}$, ϕ_n converges exactly after $n+1$ steps.

In order to tackle convergence of weakly negative formulas, we mention some general statements, where we assume that all the posets have a least element.

Lemma 18. *Convergence for* (`Roll`). *Let $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ be monotone functions. If $\mu.(f \circ g) = (f \circ g)^n(\bot)$, then $\mu.(g \circ f) = (g \circ f)^{n+1}(\bot)$.*

Lemma 19. *Convergence for* (`Diag`). *Let $f : P \times P \longrightarrow P$ be a monotone function. For each $p \in P$, put $g_p(y) = f(p,x)$ and $h(x) = \mu_y.g_x(y)$. Suppose that, for each $p \in P$, $h(p) = \mu_y.f(p,y) = g_p^n(\bot)$ and that $\mu_x.h(x) = h^m(\bot)$. Then $\mu_x.f(x,x) = f^{nm}(\bot,\bot)$.*

For our purposes, the following Lemma provides more accurate upper bounds than Lemma 19.

Lemma 20. *Let $f, g : H \longrightarrow H$ be strong monotone mappings. If $\mu.f = f^n(\bot)$ and $\mu.g = g^m(\bot)$, then $\mu.f \wedge g = (f \wedge g)^{n+m-1}(\bot)$.*

For the Bekic property we have a similar statement, bounding convergence of $\langle f, g\rangle$ by $(n+1)(m+1)-1$, with m and n being bounds on convergence of $\mu_y.g(x, y)$ and $\mu_x.f(x, \mu_y.g(x, y))$, respectively. While in general this bound is optimal, the relevant observation is, for our purposes, the following Lemma.

Lemma 21. *Let $\{x_i = f_i(x_1, \ldots, x_k) \mid i = 1, \ldots, k\}$ be a monotone system of equations P on some poset with least element $\bot$. Suppose that all the functions generated under substitution from $\{ f_1, \ldots, f_k \} \cup \{ \bot \}$ converge to their parametrized least fixed-point in one step. Then the least solution of this system of equations is obtained by iterating k times $\langle f_1, \ldots, f_k\rangle$ from $(\bot, \ldots, \bot) \in P^k$.*

Proposition 22. *Let $\phi(x)$ be a weakly negative formula, so that we have a decomposition of the form* (14). *Then $\phi(x)$ converges at its least fixed-point in at most $n + 1$ steps.*

Proof. Applying Lemma 21, we have

$$\nu.(\langle \psi_i \mid i = 1, \ldots, n\rangle \circ \psi_0) = (\langle \psi_i \mid i = 1, \ldots, n\rangle \circ \psi_0)^n(\top)\,. \tag{17}$$

Considering that

$$\mu.\phi = \mu.(\psi_0 \circ \langle \psi_i \mid i = 1, \ldots, n\rangle) = \psi_0(\nu.(\langle \psi_i \mid i = 1, \ldots, n\rangle \circ \psi_0))$$

we can use (17) and Lemma 18 to deduce that

$$\mu.\phi = (\psi_0 \circ \langle \psi_i \mid i = 1, \ldots, n\rangle)^{n+1}(\bot)\,.$$

It is possible to combine Propositions 17 and 22 with Lemma 18 to obtain upper bounds for all formulas. Yet, mainly due to the exponential blow-up in computing an equivalent normal-form of a given formula, that is, step 2 of the procedure described in Section 6, these bounds turn out to be exponential functions of the size of the formula. It is possible on the other hand to pinpoint fragments of the $\mathbf{IPC}_\mu$ for which we still have polynomial bounds. For example, if we define a formula-term to be weakly disjunctive if it is generated by the grammar (10), with the difference that we allow x to have weakly negative occurrences in α and β, then bounds are polynomials of order 2.

8. Conclusions

As mentioned in the Introduction, a main motivation for the research described in this paper was to provide in-depth answers to the question of why alternation-depth hierarchies in μ-calculi collapse or are trivial. Until now, the authors dealt with trivial alternation-depth hierarchies. The tools and ideas so far developed still need to be tested when a hierarchy does not completely collapse at its lowest level. In particular, and given the closeness of Intuitionistic Logic with Modal Logic based on transitive frames, it becomes appealing to investigate further connections with existing work on the subject [1, 2, 10, 24].

Compared to other works, such as [20, 21], we definitely took an algebraic and constructive approach to the problem of showing definability of least fixed-points within the **IPC**. Witnessing the fruitfulness of our approach, the algebra made the goal of computing upper bounds of closure ordinals of the monotone functions denoted by intuitionisitc formulas an accessible task. Let us notice on the way that our work leads to an obvious *decision procedure*, based on any decision procedure for **IPC**, for the Intuitionistic Propositional μ-Calculus. This logic, already studied on the side of proof theory and of game semantics [7], should also be of interest in verification, for example when transition systems come with some ordering and upward or downward closed properties are defined by μ-formulas, see [5].

Overall, we believe that understanding extremal fixed-points and more in general fixed-points in an intuitionisitc setting—where sparse but surprising results are known, see for [4] example—is still in quest for an elementary but solid theory to be developed. The present paper is a contribution toward this goal.

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